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A Relativity Principle in Quantum Mechanics

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Abstract

Takeuti has studied models of axiomatic set theory in which the "truth values" are elements of a complete Boolean algebra of projections on closed subspaces of a Hilbert space, and has found that the real numbers of such a model can be taken to be self-adjoint operators which can be resolved in terms of projections belonging to the Boolean algebra. It is suggested that this is the mathematical source of the replacement of real quantities by operators in quantizing a classical description, and that quantum theory involves a relativity principle with Takeuti's Boolean algebras serving as reference "frames."

1. *Introduction*

In expounding his views on the foundations of quantum mechanics, Niels Bohr[2] (probably hoping to convince Einstein of the correctness of his philosophical approach) emphasized the analogy between quantum theory and relativity theory. Bohr maintained that "the theory of relativity reminds us of the subjective character of all physical phenomena, a character which depends essentially upon the state of motion of the observer." Commenting on this in a recent work,[3] Jammer writes

> . . . Bohr erroneously generalized the relativity or reference-frame dependence of metrical attributes, such as length or duration, which in Newtonian physics are invariants, to *all* concepts of classical physics, including such invariants as rest-mass, proper-time, or charge. Bohr overlooked that the theory of relativity is also a theory of invariants and that, above all, its notion of "events" such as the collision of two particles, denotes something absolute, entirely independent of the reference frame of the observer and hence logically prior to the assignment of metrical attributes.

It is the purpose of this article to call attention to the relevance for the foundational problems in quantum theory of some recent mathematical discoveries by Gaisi Takeuti,[4] and in particular to show how these discoveries lead to an inter-

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[2] In Bohr (1929), as quoted in [2], p. 132.

[3] Jammer, (1974), p. 132.

[4] Cf. Takeuti (1975).

pretation of the formalism of quantum theory in terms of a relativity principle.

One of the curious properties of the quantum mechanical algorithms, as they are often presented, is the use of formulas from classical mechanics as intermediate steps in the derivation of quantum mechanical rules. Thus, "quantizing" a classical theory involves replacement (say in the formula for the Hamiltonian of an appropriate system) of various symbols representing real quantities by symbols representing corresponding self-adjoint operators on a Hilbert space. Takeuti discovered that in certain "nonstandard" models of axiomatic set theory, *all real variables are interpretable as self-adjoint operators on a Hilbert space.* It is this suggestive relationship that forms the basis of the interpretation suggested in this article.

Takeuti's models are particular examples of the Boolean-valued models studied by Scott and Solovay.[5] The usual models of systems formalized in predicate logic (also known as first-order functional calculus) give a value to each "sentence" of the system which is an element of the set {*true, false*}. The models studied by Scott-Solovay give values to sentences which are elements of some definite Boolean algebra (technically, the Boolean algebra must satisfy a requirement called *completeness*). Takeuti considered the particular case in which the Boolean algebra used is an algebra of projections in a Hilbert space (or equivalently, of closed subspaces of the Hilbert space). Takeuti was then able to show that, in a sense that will be explained below, the real numbers "of the model" are simply self-adjoint operators on the Hilbert space which have a spectral resolution in terms of projections belonging to the algebra.

Now, given any set of *pairwise commuting self-adjoint operators*, there is a complete Boolean algebra containing all projections needed for the spectral resolution of the given operators. Our proposal is to regard such complete Boolean algebras as *reference frames* relative to which measurements may be made of the observables corresponding to the given operators. The well-known anomalies relating to measurement of "complementary" observables then appear no more paradoxical than the relative character of measurements of space or time in special relativity. In particular the famous Einstein–Rosen–Podolsky paradox admits of a straightforward explanation. The tendency to resort to the subjectivism of the Copenhagen interpretation (according to which quantum mechanics must be understood as being not about nature but only about[6] "what we can say about nature") is now easily understood as a natural result of the lack of awareness that we inevitably perceive the quantum world only filtered through a Boolean frame. The same may be said of the proposal to regard the lattice of all projections on a Hilbert space as constituting a kind of nonclassical "quantum logic," and to take the anomalies of quantum theory as constituting an empirical refutation of ordinary logic.

Finally (and presumably this last point will ultimately prove decisive for acceptance of the interpretation being urged in this paper) the principle of relativity of quantum measurements suggests various new ways in which quantum theory can articulate with special relativity (the Lorentz group

[5] These models are described in Jech (1971).

[6] Cf. Jammer (1974), p. 204.

should be somehow combined with the unitary group to obtain a super-relativity principle) and even with general relativity (since quantizing a theory is claimed to be simply the result of applying a Boolean valuation to the sentences of the theory, this may indicate how to quantize the gravitational field).

2. *Boolean-Valued Models of Set Theory*

We shall work with the well known *language of set theory.* In this language, sentences are generated by beginning with formulas of the form $(u = v)$ and $(u \in v)$ and applying the logical operations: $\neg$ & $\vee \rightarrow \Leftrightarrow (\forall u)\ (\exists v)$, where u, v are to be selected freely from a given infinite alphabet of "variables." Despite this severely limited means of expression, it is possible, as is very well known, to express all of the propositions of ordinary mathematics in the language by using appropriate abbreviations. Moreover, the sentences representing any theorem of ordinary mathematics can be deduced from a certain familiar set of sentences, the so-called Zermelo–Fraenkel axioms (abbreviated ZF; when the axiom of choice is included the abbreviation is ZFC). We assume that the reader is familiar with these matters (see for example Jech, 1971).

In 1963, Paul Cohen developed a new and important method (called *forcing*) for constructing models of ZF and showed how models constructed in this way could be used to settle some of the most important, then outstanding, problems of axiomatic set theory (including the problems of the independence of the axiom of choice and of the continuum hypothesis). Later, Dana Scott and Robert Solovay observed that Cohen's method could be interpreted as involving a shift from the two-valued Boolean algebra (consisting of the "truth values" *true* and *false*) to certain infinite complete Boolean algebras. We begin by explaining the Scott–Solovay formulation.

Let $\mathscr{B}$ be a Boolean algebra that is complete in the sense that every nonempty set of elements of $\mathscr{B}$ has a least upper bound. We write $O, I, \vee, \wedge, \neg$, inf, sup to represent, respectively, the least element of $\mathscr{B}$, the greatest element of $\mathscr{B}$, the "join" operation on $\mathscr{B}$, the "meet" operation on $\mathscr{B}$, "complementation" on $\mathscr{B}$, the greatest lower bound of a nonempty subset of $\mathscr{B}$, and the least upper bound of a nonempty subset of $\mathscr{B}$. Using transfinite recursion, we define $V_\alpha^{(\mathscr{B})}$ for each ordinal α as follows:

$$V_0^{(\mathscr{B})} = \phi$$

$V_{\alpha+1}^{(\mathscr{B})}$ *is the set of all mappings whose domain is a subset of* $V_\alpha^{(\mathscr{B})}$ *and whose range is a subset of* $\mathscr{B}$;

$$V_\gamma^{(\mathscr{B})} = \bigcup_{\alpha<\gamma} V_\alpha^{(\mathscr{B})}$$

for γ *a limit ordinal.*

Finally we let $V^{(\mathscr{B})}$ be the collection of all elements of the $V_\alpha^{(\mathscr{B})}$ for any ordinal α. The elements of $V^{(\mathscr{B})}$ can be thought of as a kind of generalized set. Ordinary sets can be thought of as functions with values in the two-element Boolean algebra $\{0, I\}$; the elements of $V^{(\mathscr{B})}$ are functions with values in $\mathscr{B}$.

The usual techniques of elementary logic can be applied to give a "truth value" $[\![\phi]\!]$ in $\mathscr{B}$ to each sentence ϕ of our language. (Intuitively we think of the variables as ranging over $V^{(\mathscr{B})}$.) For details of the definition see Jech (1971). The result (Jech, 1971) on which all applications are based is the following:

Theorem (Scott-Solovay). Let ϕ be a sentence that can be logically deduced from the axioms ZFC. Then $[\![\phi]\!] = I$.

The Boolean-valued universe $V^{(\mathscr{B})}$ can be usefully compared with the ordinary universe V defined as follows:

$$V_0 = \phi$$

$V_{\alpha+1}$ is the set of all subsets of V_α

$$V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha \text{ for } \gamma \text{ a limit ordinal}$$

V is the collection of all elements of the V_α.

We define a map $x \to \check{x}$ which embeds V in $V^{(\mathscr{B})}$ by recursion as follows:

For each $v \in V$, $\check{v}$ is the map with constant value I defined on $\{\check{u} \mid u \in v\}$.

An element $a \in U$ is called *absolute* if for some formula $\phi(x)$ the sentences $\phi(a)$ and

$$(\forall x, y)\,[(\phi(x)\ \&\ \phi(y)) \to (x = y)]$$

are provable from ZFC and

$$[\![\phi(\check{a})]\!] = I$$

(In such a case we say that "a in $V^{\mathscr{B}}$ is $\check{a}$".) In particular, if $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers, then for each $n \in \omega$, n in $V^{\mathscr{B}}$ is $\check{n}$. Also ω in $V^{\mathscr{B}}$ is $\check{\omega}$. If $\mathbb{Q}$, the set of rational numbers is defined as usual, then for each $r \in \mathbb{Q}$, r in $V^{\mathscr{B}}$ is $\check{r}$ and also $\mathbb{Q}$ in $V^{\mathscr{B}}$ is $\check{\mathbb{Q}}$. When no ambiguities result, we feel free to omit the "V" in such cases. What is interesting for us is that if $\mathbb{R}$ is the set of real numbers then $\mathbb{R}$ in $V^{\mathscr{B}}$ need not be $\check{\mathbb{R}}$.

3. *Algebras of Projections*

Let $\mathscr{H}$ be a Hilbert space. A bounded linear operator P on $\mathscr{H}$ is called a projection if $P^2 = P$ and P is self-adjoint. In such a case the range of P is a closed linear subspace L of $\mathscr{H}$ and P is the (orthogonal) projection operator on L. (I.e., for $x \in \mathscr{H}$, Px is the point of L nearest to x in the Hilbert space metric.) In particular, 0 is the projection on the zero-dimensional subspace of $\mathscr{H}$ consisting of the origin, and the identity I is the projection on the entire space $\mathscr{H}$. If P and Q are the projections on L and M, respectively, then we may define the Boolean operations $\neg P$, $P \vee Q$, $P \wedge Q$. Namely, $\neg P$ is the projection on the orthogonal complement of L, $P \vee Q$ is the projection on the smallest linear subspace containing L and M, and $P \wedge Q$ is the projection on $L \cap M$. A Boolean algebra of projections is simply a set of projections containing 0 and I and closed under the three Boolean operations. Such an algebra $\mathscr{B}$ is *complete* if, whenever $P_\alpha \in \mathscr{B}$ for all $\alpha \in J$ where each P_α is the projection on L_α, the projection on the smallest *closed* subspace of $\mathscr{H}$ containing all of the P_α is likewise in $\mathscr{B}$. Such an algebra is a complete Boolean algebra in the usual sense, and all of the considerations of the preceding section apply.

From now on, let $\mathscr{B}$ be a complete algebra of projections on $\mathscr{H}$. Then $(\mathscr{B})$ is the set of all self-adjoint operators A such that if we write the spectral resolution

$$A = \int \lambda \, dE_\lambda$$

of A, then each projection E_λ belongs to $\mathscr{B}$. The key fact on which our interpretation of quantum mechanics is based is that given a set $\{A_\alpha\}$ of self-adjoint operators on $\mathscr{H}$, *there exists a complete Boolean algebra $\mathscr{B}$ of projections for which $\mathscr{B}$ contains all of the A_α, if and only if each pair A_α, A_β commute.*

A set of elements P_α of $\mathscr{B}$ is called a *partition of unity* if

$$P_\alpha \cdot P_\beta = 0 \text{ unless } \alpha = \beta \tag{1}$$

$$\sum_\alpha P_\alpha = I \tag{2}$$

where the sum is understood in the sense of the strong topology on $\mathscr{H}$. Now, we can show that $[\![u \in \mathbb{Q}]\!] = I$ for $u \in V^{\mathscr{B}}$ if and only if u can be represented by

$$u = \sum_i q_i P_i$$

where each $q_i \in \mathbb{Q}$ and where $\{P_i\}$ is a partition of unity. Now, let us write

$$\mathbb{R}^{\mathscr{B}} = \{u \in V^{\mathscr{B}} \mid [\![u \in \mathbb{R}]\!] = I\}$$

where $\mathbb{R}$ is defined as the set of upper Dedekind sections on $\mathbb{Q}$. Then for any $u \in \mathbb{R}^{\mathscr{B}}$ we can define for any real number λ

$$E_\lambda = \inf_{r > \lambda} [\![r \in u]\!]$$

Then

$$A = \int \lambda \, dE_\lambda \tag{*}$$

is well defined and is a self-adjoint operator on $\mathscr{H}$. Conversely, given a self-adjoint operator $A \in (\mathscr{B})$, we can write its spectral resolution (*) and then define

$$u(r) = E_r$$

for each $r \in \mathbb{Q}$. Then $u \in \mathbb{R}^{\mathscr{B}}$. The one-to-one correspondence just defined between $\mathbb{R}^{\mathscr{B}}$ and $(\mathscr{B})$ can be shown to preserve +, ·, and $\leqslant$. It is in this sense that the real numbers of our model are self-adjoint operators on $\mathscr{H}$.

4. *Boolean Frames in Quantum Mechanics*

As in special relativity theory, we assume that *all measurements are relative to the frame of observation.* In special relativity the appropriate kind of frame is the so-called *inertial frame.* In quantum theory, it is a complete Boolean algebra of projections.

Quantities that can be measured simultaneously correspond to commuting self-adjoint operators; hence there is a complete Boolean algebra $\mathscr{B}$ of projections such that $(\mathscr{B})$ contains all such operators. Such an algebra $\mathscr{B}$ is a reference frame with respect to which the measurements are being made. The secret of complementarity is then simply that, for a pair of complementary quantities, there is no Boolean algebra $\mathscr{B}$ such that $(\mathscr{B})$ contains both of them. An interaction corresponding to a position measurement and one corresponding to a momentum measurement are thus measurements with respect to distinct frames of reference (as in relativity theory are measurements of space-time coordinates with respect to observers moving relative to one another).

The underlying conceptual framework is as follows: We suppose that we have given a collection of sentences of the language of set theory which express various relationships among real physical quantities. Some of these sentences express the basic physics. Others express properties of specific physical systems including measurement apparatus. In particular, some sentences may express the result of a measurement. The correctness of the description of reality given by these sentences ϕ can be stated as

$$[\![\phi]\!] = I$$

for any complete Boolean algebra $\mathscr{B}$. The role of a particular Boolean frame can be seen if we analyze a sentence that expresses the fact that a real quantity that is the result of an interaction with some apparatus satisfies some condition (e.g., an inequality). Such a sentence has the form $p \Rightarrow q$ when p expresses the effect of the interaction and q the resulting condition. (For example, if a particle's presence in a slit—a "position" measurement—is revealed by the action of a counter, we may take p to be the statement "the counter clicks" and q the statement "the particle was in the slit.") We will have $[\![p \Rightarrow q]\!] = I$ in every Boolean frame. In order to obtain from this the desired $[\![q]\!] = I$, we will need a frame in which $[\![p]\!] = I$. The physics of the apparatus determines an appropriate frame in which this last holds.

Note that since the transition from real quantities to self-adjoint operators is with respect to a particular frame, the fact that a particular observable is represented by some specific operator also must be taken relative to a frame.

We conclude this section by referring to Table I which indicates by a comparison with special relativity how quantum theory is to be regarded as a relativity theory. So far we have only mentioned the analogy between inertial frames and Boolean frames. In special relativity theory, the Lorentz transformation gives the mathematical relationship between measurements made from certain corresponding frames. One example of such a transformation in quantum theory is the Fourier transform, which establishes a similar correspondence

TABLE I

Special relativity	Quantum theory
Inertial frame	Boolean frame
Lorentz transformation	Fourier transform
Lorentz group	Unitary group (or appropriate subgroup)
Space-time vector dx^{μ}	State vector ψ
Minkowski metric $dx^2 + dy^2 + dz^2 - dt^2$	Hilbert space metric $\psi^*\psi$

between "position space" and "momentum space". Since both the Fourier transform and the Schrödinger transform

$$U(t) = e^{-itH}$$

are unitary, the analog of the Lorentz group is some subgroup of the unitary group (perhaps the entire unitary group). Finally the state vector ψ is an invariant independent of the frame (hence analogous to the Minkowski metric).

5. *The Anomalies of Quantum Mechanics*

5.1. *The two-slit experiment.* There are two slits in front of a screen containing counters. A beam of particles is prepared that can reach the screen only by passing through one of the slits. The anomaly may be expressed by the fact that the pattern on the screen when both slits are open is not the union of the patterns obtained when each is open separately. (The anomaly persists even when the beam is of such low intensity that at most one particle can be in the vicinity of the slits at any given time.) Our explanation is simply that opening both slits involves measuring relative to a Boolean frame (one corresponding to a momentum measurement) different from the frame involved when only one slit is open (which corresponds to a position measurement).

5.2. *Schrödinger's Cat.* An apparatus is designed which amplifies a microscopic event of probability 1/2 (say, a photon passing through a half-silvered mirror) to a decidedly macroscopic one—the electrocution of a cat. (Why Schrödinger chose this relatively innocent species for his example, when he could have chosen an example more worthy, such as the mosquito, must remain a mystery.) If all this happens in a closed box, is it the opening of the lid by an observer which causes the state function which assigns equal weights of life and death to the perhaps unfortunate animal to suddenly collapse into one or another of the eigenstates? We respond: there is no collapse. The Boolean frame corresponding to Schrödinger's macabre "measurement" will contain a pair of projections corresponding to "the cat lives" and "the cat is dead." Opening the lid of the box has no particular effect.

5.3. *The EPR Paradox.* Two particles A and B interact in such a way that the sum of their momenta after the interaction is 0. Hence a measurement of the momentum of A will also yield the momentum of B. Such a measurement thus will not only cause a "collapse" of the state function of A, but also that of B, which by this time may be very far away.

Our analysis is quite simple. Let p_A, p_B represent the momenta of A and B, respectively, at some given time t. Choosing an appropriate frame:

$$[\![(p_A + p_B) = 0]\!] = I$$

By Takeuti's work this gives

$$M_A + M_B = 0$$

where M_A, M_B are the momentum operators corresponding to A and B, respectively.

6. *Comparison with Everett's "Many Worlds" Interpretation*

Our interpretation has in common with Everett's "many worlds" interpretation[7] that the state vector does not collapse into an eigenstate as the result of a measurement. But the views are really quite different. We have no need to envision a multiplicity of universes. The difference is seen quite clearly by contrasting our treatment of Schrödinger's cat paradox with Everett's. In Everett's view, carrying out Schrödinger's experiment causes the universe to split in two: in one world the cat is alive and well; in the other it is, alas, a charred corpse.

7. *Suggestions for Further Work*

One has every right to expect that the advantage of possessing the "correct" point of view on a foundational issue will be made clear by the scientific progress that results. Here we want to suggest two directions for research that our interpretation suggests:

(a) If quantum theory embodies a relativity principle, it surely interacts with the other basic relativity principles. Thus, there should be a relativity theory combining special relativity and quantum theory in which the underlying group combines (perhaps as a direct product) the Lorentz and the unitary groups.

(b) If "quantizing" a classical theory simply involves applying appropriate Boolean valuations to the sentences of the theory, the same process should work for quantum field theory, and even for the quantization of the gravitational field.

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[7] Cf. Jammer (1974), pp. 507–521.